Lecture 6: Exploring Regular Surfaces

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Math 142:
Differential Geometry

Two Shortcuts

The last example in the previous lecture shows that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome.

Shortcut 1

If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface

Proof.

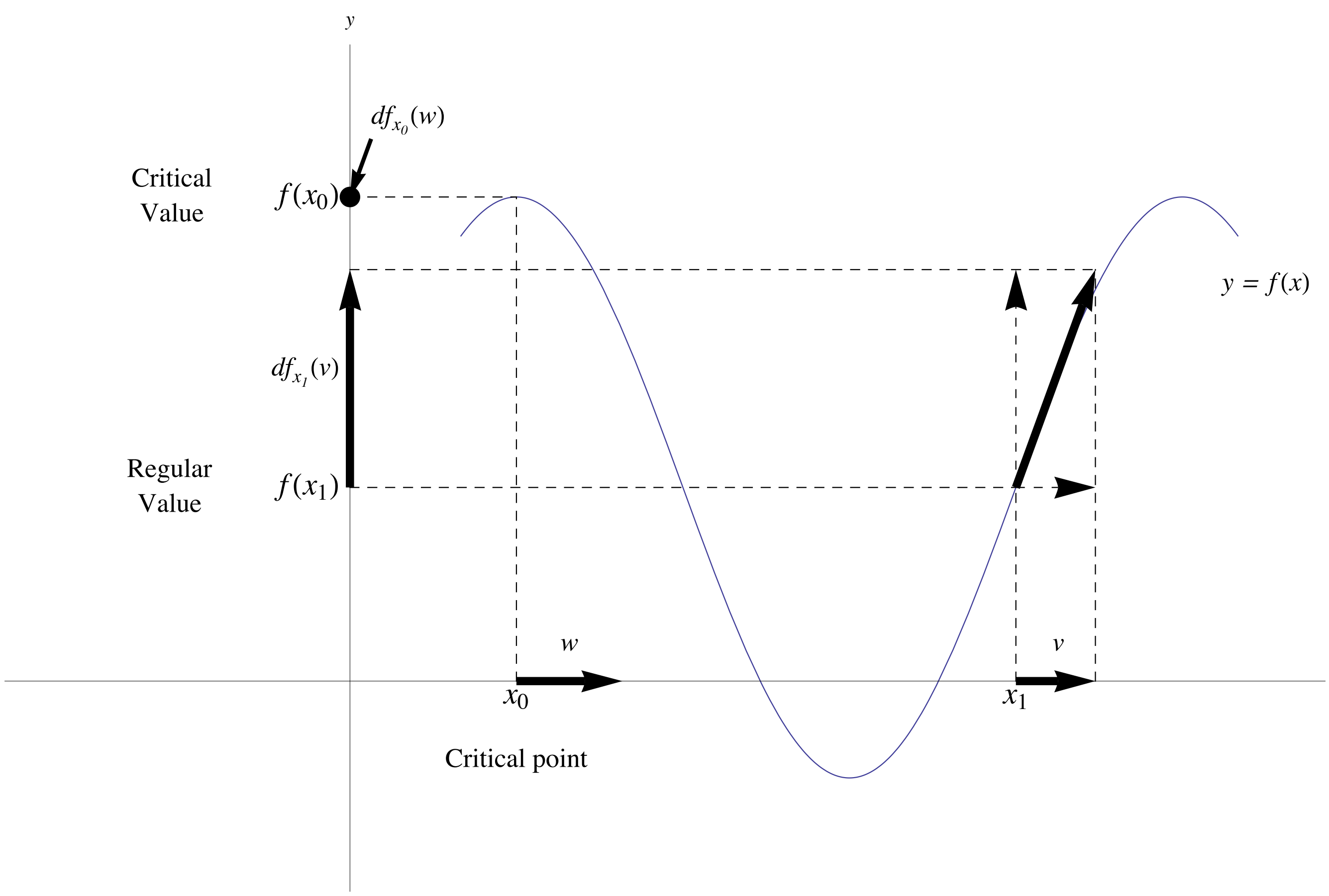
$\square$

Critical Points and Values

Definition

Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^m$ which is not a critical value is called a *regular value* of F.

The terminology is evidently motivated by the particular case in which $f : U \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function of a real variable. A point $x_0 \in U$ is critical if $f'(x_0) = 0$, that is, if the differential df_{x_0} carries all the vectors in $\mathbb{R}$ to the zero vector. Notice that any point $a \notin f(U)$ is trivially a regular value of f.


y
$df_{x_0}(w)$
Critical
Value
$f(x_0)$
$y = f(x)$
$df_{x_1}(v)$
Regular
Value
$f(x_1)$
w
v
x_0
x_1
Critical point

Critical Points and Values

Remark

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then

$$df_p = (f_x, f_y, f_z).$$

Note, in this case, that to say that df_p is not surjective is equivalent to saying that $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ if and only if f_x, f_y, and f_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}.$$

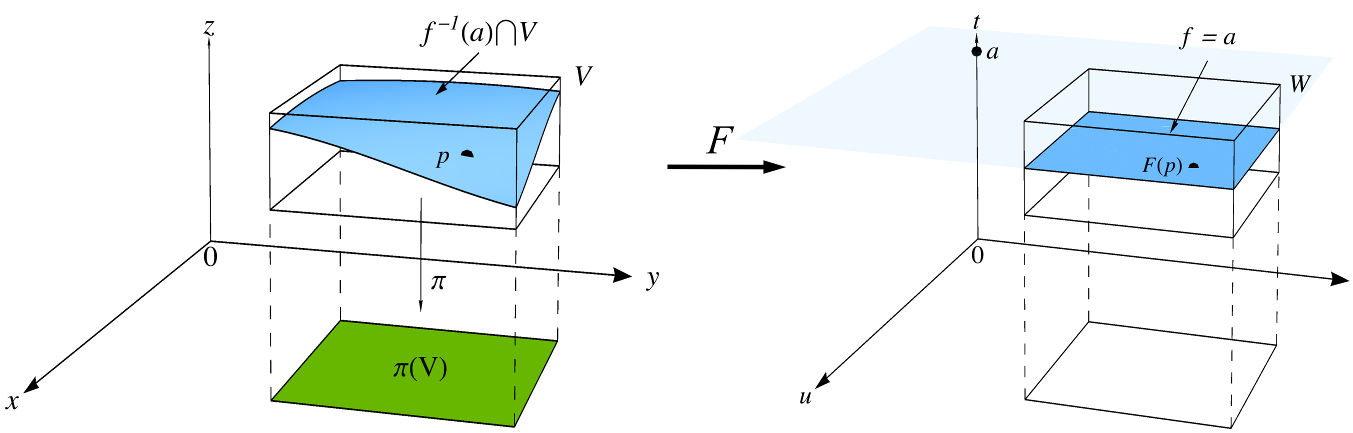
Two Shortcuts

Shortcut 2

If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Proof

Let $p = (x_0, y_0, z_0)$ be a point of $f^{-1}(a)$. Since a is a regular value of f, it is possible to assume, by renaming the axis if necessary, that $f_z \neq 0$ at p.



Examples

Example

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

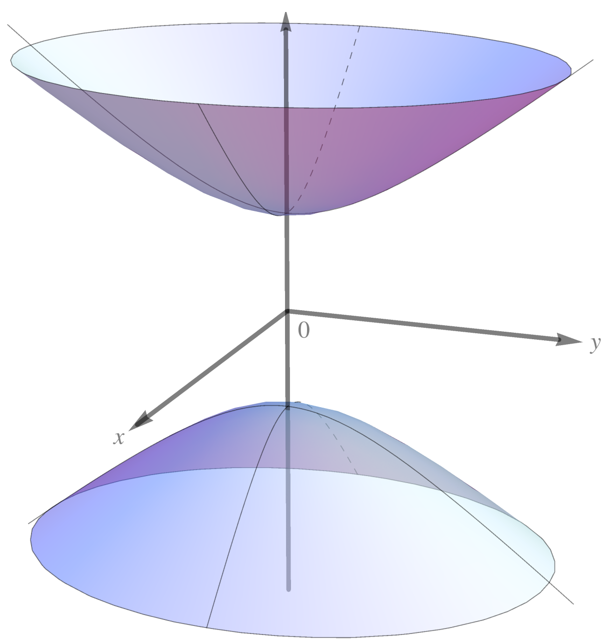
is a regular surface.

The examples of regular surfaces presented so far have been connected subsets of $\mathbb{R}^3$. A surface $S \subset \mathbb{R}^3$ if said to be *connected* if any two of its points can be joined by a continuous curve in S. In the definition of a regular surface we made no restrictions on the connectedness of the surfaces, and the following example shows that the regular surfaces given by Shortcut 2 may not be connected.

Examples

Example

The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface. Note that the surface S is not connected.



Examples

Example

The torus T is a "surface" generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle.

Proof

Let S^1 be the circle in the yz plane with its center on the point $(0, a, 0)$. Then S^1 is given by $(y - a)^2 + z^2 = r^2$.

The points of T are obtained by rotating this circle about the z axis satisfying the equation

$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = r^2.$$

Examples

Proof (cont'd)

Let $f(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2$. Then

$$\frac{\partial f}{\partial z} = 2z, \quad \frac{\partial f}{\partial y} = \frac{2y\left(\sqrt{x^2 + y^2} - a\right)}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial x} = \frac{2x\left(\sqrt{x^2 + y^2} - a\right)}{\sqrt{x^2 + y^2}}.$$

Hence, $(f_x, f_y, f_z) \neq (0, 0, 0)$ in $f^{-1}(r^2)$, so r^2 is a regular value. Therefore, the torus is a regular surface.

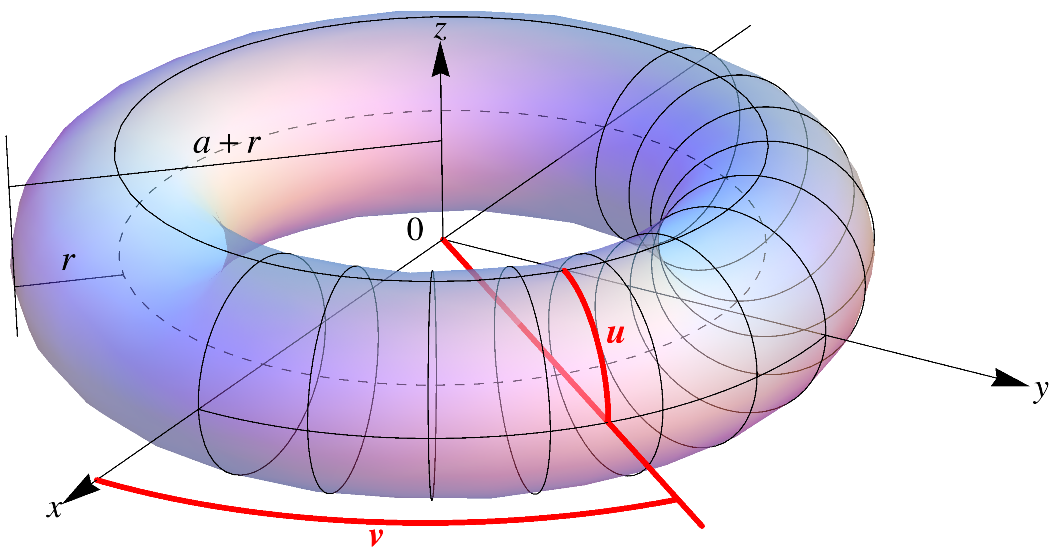
Examples

Example

A parametrization for the torus T of the previous example can be given by

$$\mathbf{x}(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.



A Very Useful Fact

Fact

If $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ *is a nonzero continuous function defined on a connected surface* S*, the* f *does not change sign on* S*.*

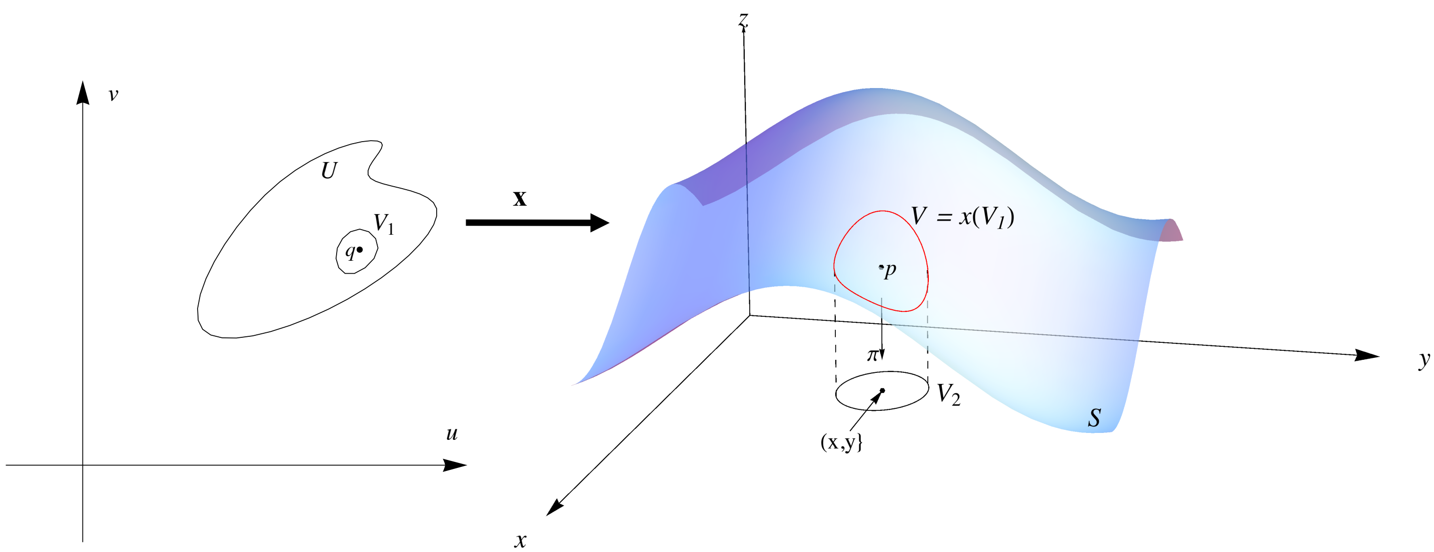
Proof.

□

Two Propositions

Proposition

Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. (This proposition is usually used to prove that a subset of $\mathbb{R}^3$ is not a regular surface.)



Two Propositions

Proposition

Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subset S$ such that conditions 1 and 3 of the definition hold. Assume that $\mathbf{x}$ is one-to-one. Then $\mathbf{x}^{-1}$ is continuous.